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CONTACT INTERACTION WITH FRICTION OF TWO ELASTIC WEDGE-SHAPED BODIES OF DIFFERENT MATERIALS*

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The plane static contact problem of elasticity theory concerning the impression of one wedge-shaped body into another of different material along sections of the side surfaces is examined. The abutting sections of both wedges start from the vertices. The problem is solved taking friction into account. In the case of greatest interest for applications, when the aperture angle of one of the wedges is π, an exact closed solution is constructed in the form of Cauchy-type integrals. However, the method of solution can be used for any wedge aperture angle.

1. Formulation of the problem. We consider two elastic wedge-shaped bodies of different materials, one being impressed into the other along sections of the side surfaces. The contact sections in both wedges start from the vertices (Fig.1). Outside the line of contact the wedge faces are stress-free. It is assumed that the length of the line of contact is small compared with the characteristic dimensions of both bodies.

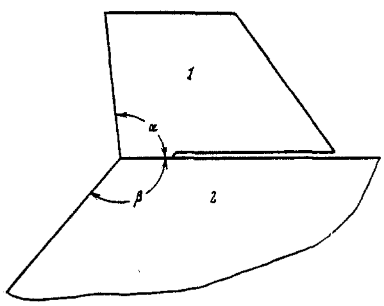

Fig.1

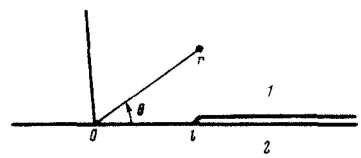

Fig. 2

Applying the "microscope principle" /1, 2/, we arrive at a singular boundary value problem whose boundary conditions have the form (Fig.2)

$$\theta = \alpha,\ \sigma_\theta = \tau_{r\theta} = 0;\ \theta = -\pi,\ \sigma_\theta = \tau_{r\theta} = 0 \tag{1.1}$$

$$\theta = 0,\ [\sigma_\theta] = [\tau_{r\theta}] = 0,\ \tau_{r\theta} = -k\sigma_\theta$$

$$\theta = 0,\ r < l,\ [u_\theta] = f(r);\ \theta = 0,\ r > l,\ \sigma_\theta = 0 \tag{1.2}$$

$$\int_0^l \sigma_\theta(r, 0)\, dr = Y \tag{1.3}$$

$$r \to \infty,\ V \equiv (\sigma_\theta, \tau_{r\theta}, \sigma_r) = O(1/r)$$

Here r, θ are polar coordinates, σ_θ, $\tau_{r\theta}$, σ_r are stresses, u_θ, u_r are displacements, $[N]$ is the jump in the quantity N, $k > 0$ is the coefficient friction, $f(r)$ is a given function, $(-kY, Y)$ is the given principal vector of the forces in the section $\theta = 0,\ 0 < r < l$.

The aperture angle of one of the wedges is taken equal to π since it is this case that is of greatest interest in connection with possible applications of the solution constructed below in engineering problems of material treatment and fracture (for instance, when cutting metals). The problem under consideration can here be treated exactly as a problem on the motion of an elastic wedge with friction over the surface of a half-space from another elastic material. However, the method described later for the solution is suitable for any wedge aperture angle (and α and β in Fig.1).

Similar problems were examined in /3/. Their solutions were constructed in the form of infinite products.

As $r \to 0$ the solution of the problem under consideration behaves as the greatest solution asymptotically satisfying the condition of continuity of the displacements at an angular

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point for the canonical singular problem with boundary conditions

$$\theta = \alpha, \quad \sigma_\theta = \tau_{r\theta} = 0; \quad \theta = -\pi, \quad \sigma_\theta = \tau_{r\theta} = 0$$
$$\theta = 0, \quad [\sigma_\theta] = [\tau_{r\theta}] = 0, \quad \tau_{r\theta} = -k\sigma_\theta, \quad [u_\theta] = 0$$

The solution mentioned for the canonical singular problem is constructed by the method of singular solutions /1, 2/, and should be realized as the asymptotic form of the desired solution of the initial problem as $r \to 0$. Therefore, the following asymptotic forms hold for the stresses near the angular point in the initial problem for $0 \leqslant \theta \leqslant \alpha$, $r \to 0$

$$\begin{aligned}\sigma_\theta \sim C_0 \{&[\lambda \sin(\lambda+2)\theta - (\lambda+2)\sin\lambda\theta]\Delta_+ - \\ &k(\lambda+2)[\cos\lambda\theta - \cos(\lambda+2)\theta]\Delta_- - \\ &2[k(\lambda+2)\sin\lambda\theta + \lambda\cos(\lambda+2)\theta]\delta_+ + \\ &2(\lambda+2)[\cos\lambda\theta + k\sin(\lambda+2)\theta]\delta_-\}\end{aligned} \tag{1.4}$$

$$\begin{aligned}\tau_{r\theta} \sim C_0 \{&\lambda[\cos\lambda\theta - \cos(\lambda+2)\theta]\Delta_+ - \\ &k[\lambda\sin\lambda\theta - (\lambda+2)\sin(\lambda+2)\theta]\Delta_- - \\ &2\lambda[\sin(\lambda+2)\theta - k\cos\lambda\theta]\delta_+ + \\ &2[\lambda\sin\lambda\theta - k(\lambda+2)\cos(\lambda+2)\theta]\delta_-\}\end{aligned}$$

$$\begin{aligned}\sigma_r \sim C_0 \{&[(\lambda-2)\sin\lambda\theta - \lambda\sin(\lambda+2)\theta]\Delta_+ - \\ &k[(\lambda+2)\cos(\lambda+2)\theta - (\lambda-2)\cos\lambda\theta]\Delta_- + \\ &2[\lambda\cos(\lambda+2)\theta + k(\lambda-2)\sin\lambda\theta]\delta_+ - \\ &2[k(\lambda+2)\sin(\lambda+2)\theta + (\lambda-2)\cos\lambda\theta]\delta_-\}\end{aligned}$$

$$C_0 = \frac{C(2\pi r)^\lambda}{8\lambda\sin\pi\lambda}, \quad \Delta_\pm = \sin 2(\lambda+1)\alpha \pm (\lambda+1)\sin 2\alpha$$

$$\delta_\pm = \sin^2(\lambda+1)\alpha \pm (\lambda+1)\sin^2\alpha$$

The asymptotic forms for $-\pi \leqslant \theta \leqslant 0$, $r \to 0$ have a form analogous to (1.4) when $\Delta_\pm$ is replaced by $\cos\pi\lambda$, and $2\delta_\pm$ by $-\sin\pi\lambda$ and C_0 by

$$-\frac{2C_0\delta}{\sin\pi\lambda}, \quad \delta = \sin^2(\lambda+1)\alpha - (\lambda+1)^2\sin^2\alpha$$

Here C is a real constant with the dimensions of a force, divided by the length to the power $\lambda+2$, which is defined below from the solution constructed for the initial problem, $\lambda = \lambda(\alpha, k, E, \nu_1, \nu_2)$ is the single root of the characteristic equation

$$\Delta_+ \sin\pi\lambda + 2En\delta\cos\pi\lambda + 2k(\lambda+1)(\lambda+2)\sin\pi\lambda\sin^2\alpha + 4Uk\delta\sin\pi\lambda = 0$$

$$\left(E = \frac{E_1}{E_2}, \quad n = \frac{1-\nu_2^2}{1-\nu_1^2}, \quad U = \frac{1-2\nu_1}{4(1-\nu_1)} - \frac{1-2\nu_2}{4(1-\nu_2)}En\right)$$

in the interval $-1 < \lambda < 0$ (E_1, E_2 and ν_1, ν_2 are Yong's moduli and Poisson's ratios of materials 1 and 2).

For those values of $\alpha, k, E, \nu_1, \nu_2$ for which the equation has no roots in the interval mentioned, the stresses in the initial problem are bounded as $r \to 0$.

Values of the quantity $(\lambda+1)\cdot 10^3$ are presented in the table for $\nu_1 = 0.250$ and $\nu_2 = 0.333$. The empty cells denote that the characteristic equation has no roots in the interval $-1 < \lambda < 0$ for corresponding value sof the parameters.

The asymptotic form of the desired solution as $r \to l$ is obtained from the preceding one for $\alpha = \pi$ if the subscripts 1 and 2 are interchanged. In particular,

$$\frac{E_1}{4(1-\nu_1^2)}\left[\frac{\partial u_\theta}{\partial r}\right]\Big|_{\theta=0} \sim (1+En)R[2\pi(r-l)]^{-\gamma/\pi} \quad (r \to l+0) \tag{1.5}$$
$$\sigma_\theta(r, 0) \sim (1+En)Rq[2\pi(l-r)]^{-\gamma/\pi} \quad (r \to l-0)$$
$$R = {}^1/_4(\pi/\gamma)K, \quad \gamma = \arccos(-qkU), \quad q = 2[(1+En)^2 + 4k^2U^2]^{-1/2}$$

Here K is a coefficient with the dimensions of force, divided by length to the power $2 - \gamma/\pi$ and to be determined.

If the length l of the line of contact is unknown (for a smooth function $f(r)$), then it is determined from the condition that the coefficient K equals zero.

2. Solution of the Wiener-Hopf equation. Applying the Mellin integral transform to the equilibrium equations, the strain compatibility condition, Hooke's law, the "through" conditions (1.1), and taking account of the "dual" conditions (1.2), we arrive at the Wiener-Hopf functional equation for the problem under consideration

$$\Phi^-(p) = G_0(p)[\Phi^+(p) + g(p)] \quad (-\lambda - 1 < \operatorname{Re} p < 0) \tag{2.1}$$

$$\Phi^+(p) = \frac{E_1}{4(1-\nu_1^2)}\int_1^\infty \left[\frac{\partial u_\theta}{\partial r}\right]\Big|_{\substack{r=\rho l\\ \theta=0}} \rho^p \, d\rho$$

$$\Phi^-(p) = \int_0^1 \sigma_\theta(\rho l, 0)\rho^p\,d\rho, \quad g(p) = \frac{E_1}{4(1-\nu_1^2)}\int_0^1 f'(\rho l)\rho^p\,d\rho$$

$$G_0(p) = -4d(p)\sin p\pi/\Delta(p), \quad d(p) = \sin^2 p\alpha - p^2\sin^2\alpha$$

$$\Delta(p) = (\sin 2p\alpha + p\sin 2\alpha)\sin p\pi + 2End(p)\cos p\pi -$$
$$2kp(p-1)\sin p\pi\sin^2\alpha - 4Ukd(p)\sin p\pi$$

(In cases when the characteristic equation, presented in the preceding section, has no roots in the interval $(-1, 0)$, we obtain $\lambda = 0$).

E	$\alpha=10°$	30°	50°	70°	90	110°	130°	150°	180°
			$k=0.001$						
10^{-4}						698	563	512	500
10^{-2}						696	563	512	500
10^{-1}						682	557	511	500
1					781	606	532	507	500
10				661	558	521	506	501	500
10^2		804	563	519	507	502	500	500	500
10^4	595	504	500	500	500	500	500	500	500
			$k=0.1$						
10^{-4}						761	596	530	511
10^{-2}						758	594	529	511
10^{-1}						738	586	527	509
1					839	632	545	511	502
10				674	561	519	502	496	494
10^2		813	560	514	500	495	493	493	493
10^4	589	496	493	493	492	492	492	492	492
			$k=1$						
10^{-4}								706	603
10^{-2}								703	601
10^{-1}								679	588
1							679	561	516
10				836	593	504	464	446	439
10^2			532	463	440	431	426	425	424
10^4	532	427	423	422	422	423	422	422	422

The functions $\Phi^-(p), \Phi^+(p)$ in (2.1) are analytic, respectively, in the half-planes $\operatorname{Re} p > -\lambda - 1$, $\operatorname{Re} p < 0$. Using the asymptotic form (1.5), we obtain by a theorem of Abelian type /4/

$$p \to \infty, \ \Phi^-(p) \sim qZp^{\gamma/\pi - 1}, \ \Phi^+(p) \sim Z(-p)^{\gamma/\pi - 1} \tag{2.2}$$
$$Z = R(1 + En)\Gamma(1 - \gamma/\pi)(2\pi l)^{-\gamma/\pi}$$

($\Gamma(z)$ is the Gamma function).

We rewrite (2.1) in the following form:

$$\Phi^-(p) = -\frac{q\sin p\pi}{\sin(p\pi+\gamma)}G(p)[\Phi^+(p) + g(p)] \quad (-\lambda - 1 < \operatorname{Re} p < 0) \tag{2.3}$$
$$G(p) = 2[(1+En)\cos p\pi - 2Uk\sin p\pi]\,d(p)/\Delta(p)$$

The function $\operatorname{Re} G(it)$ $(-\infty < t < \infty)$ is a positive even function of t that tends to unity as $t \to \infty$, while the function $\operatorname{Im} G(it)$ $(-\infty < t < \infty)$ is an odd function of t that tends to zero as $t \to \infty$. Therefore, the index of the function $G(p)$ along the imaginary axis equals zero and the following factorization holds /5/:

$$G(p) = \frac{G^+(p)}{G^-(p)} \quad (\operatorname{Re} p = 0) \tag{2.4}$$
$$\exp\left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln G(\tau)}{\tau - p}d\tau\right] = \begin{cases} G^+(p), & \operatorname{Re} p < 0 \\ G^-(p), & \operatorname{Re} p > 0 \end{cases}$$

We use the following representation

$$\frac{p + \gamma/\pi}{p}\frac{\sin p\pi}{\sin(p\pi+\gamma)} = K^+(p)K^-(p), \quad K^\pm(p) = \frac{\Gamma(1 \mp p \mp \gamma/\pi)}{\Gamma(1 \mp p)} \tag{2.5}$$

The functions $K^-(p), K^+(p)$ are analytic and have no zeros in the half-planes $\operatorname{Re} p > -1$, $\operatorname{Re} p < 1 - \gamma/\pi$, respectively. Moreover, the following asymptotic forms hold:

$$p \to \infty, \ K^+(p) \sim (-p)^{-\gamma/\pi}, \ K^-(p) \sim p^{\gamma/\pi} \tag{2.6}$$

Using the factorizations (2.4) and (2.5), and the representation

$$K^+(p)G^+(p)g(p) = g^+(p) - g^-(p) \quad (\operatorname{Re} p = 0) \tag{2.7}$$
$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}K^+(\tau)G^+(\tau)g(\tau)\frac{d\tau}{\tau - p} = \begin{cases} g^+(p), & \operatorname{Re} p < 0 \\ g^-(p), & \operatorname{Re} p > 0 \end{cases}$$

we write the functional equation (2.3) thus

$$(p + \gamma/\pi)\,[K^-(p)]^{-1}\,\Phi^-(p)\,G^-(p) - qpg^-(p) = \\ -qpK^+(p)\,\Phi^+(p)\,G^+(p) - qpg^+(p)\ (\operatorname{Re} p = 0) \tag{2.8}$$

The functions on the left and right sides of (2.8) are analytic, respectively, in the half-planes $\operatorname{Re} p > 0, \operatorname{Re} p < 0$. By the principle of analytic continuation they equal the same function that is analytic in the whole p plane. It follows from (2.2), (2.4), (2.6), (2.7) that the functions on the left and right sides of (2.8) tend to the constant

$$a = q(Z - \delta), \quad \delta = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K^+(p)\,G^+(p)\,g(p)\,dp$$

as $p \to \infty$. By Liouville's theorem, a single analytic function is identically equal to this constant in the whole p plane

Taking (1.3) into account, we find

$$a = \frac{G^-(0)}{\Gamma(\gamma/\pi)}\frac{Y}{l} \tag{2.9}$$

The solution of the functional equation (2.1) has the form

$$\Phi^+(p) = -[qpK^+(p)\,G^+(p)]^{-1}\,[a + qpg^+(p)]\ \ (\operatorname{Re} p < 0) \\ \Phi^-(p) = (p + \gamma/\pi)^{-1}\,K^-(p)\,[G^-(p)]^{-1}\,[a + qpg^-(p)]\ (\operatorname{Re} p > 0) \tag{2.10}$$

3. The coefficients K and C. Formula for the contact stress. We find from the equation $q(Z - \delta) = a$, in which a is given by (2.9),

$$K = \frac{8\gamma(2\pi)^{\gamma/\pi - 1}}{q(1 + En)\,\Gamma(1 - \gamma/\pi)}\ \ (a + q\delta)\,l^{\gamma/\pi} \tag{3.1}$$

For a smooth function $f(r)$ we must determine the length l of the contact area. Since $K = 0$ in this case, then according to (2.9) and (3.1),

$$l = -\frac{G^-(0)}{q\Gamma(\gamma/\pi)}\frac{Y}{\delta}$$

Using (2.1) and (2.10) we obtain the following formula for the contact stress $(\theta = 0, 0 < r < l)$:

$$\sigma_\theta = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{4d(p)\sin p\pi}{\Delta(p)}\{[qpK^+(p)\,G^+(p)]^{-1} \times [a + qpg^+(p)] - g(p)\}\left(\frac{r}{l}\right)^{-p-1} dp \tag{3.2}$$

Using (3.2) and the formula for the stress near the point O presented in Sect.1, we find the coefficient C characterizing the behaviour of the stress at an angular point

$$C = \frac{8\lambda\sin^2\pi\lambda}{(2\pi)^\lambda\,\Delta'(-\lambda - 1)} \times \left[\frac{q(\lambda + 1)\,g^+(-\lambda - 1) - a}{q(\lambda + 1)K^+(-\lambda - 1)\,G^+(-\lambda - 1)} - g(-\lambda - 1)\right] l^{-\lambda}$$

($\Delta'(p)$ is the derivative of the function $\Delta(p)$ with respect to p).

REFERENCES

1. CHEREPANOV G.P., On singular solutions in elasticity theory. Problems of the Mechanics of a Solid Deformed Body. Sudostroenie, Leningrad, 1970.
2. CHEREPANOV G.P., Mechanics of Brittle Fracture. McGraw-Hill, New York, 1979.
3. DYKHNOV A.E., Contact problem for elastic wedges when there is friction along parts of the faces adjacent to a common apex, Dokl. Akad. Nauk SSSR, Vol.249, No.4, 1979.
4. NOBLE B., Application of the Wiener-Hopf Method to Solve Partial Differential Equations /Russian translation/, Izd. Inostr. Lit., Moscow, 1962.
5. GAKHOV F.D., Boundary Value Problems, Nauka, Moscow, 1977.

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